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A Modification of the HP Filter

Aiming at Reducing the End-Point Bias

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Abstract

It is well known that the Hodrick-Prescott (HP) filter suffers from an end-point bias. This is problematic when the filter is used recursively for economic policy (in this case the end-point is the point of interest). The usual way to deal with this problem is to extend the series with ARIMA forecasts. However, the usefulness of this approach is limited by the quality of the forecast. This is why the present paper explores an alternative way to deal with the end-point bias which does not use forecasts: the penalty function that the HP filter minimizes is modified in order to reduce (however, without eliminating) the difference in treatment of the end-point compared to other points; this yields a modified HP filter. It is shown that (compared to the usual HP filter) the modified HP filter (recursively applied) has the following properties. i) The end-point bias is reduced. ii) The business cycle component is bigger; the modified HP filter is in fact approximately equivalent to another modification of the HP filter: increasing the gap between trend and data by a constant (43% when the smoothing parameter is 100). iii) The amplitude response of the modified HP filter is closer to the one of the ideal filter, but there is a phase shift.

1. Introduction

The Hodrick-Prescott (HP) filter[1] is often used to separate structural and cyclical components, or to smooth a curve (without smoothing it to the point that it becomes a straight line). However, this filter has been criticized on several grounds, in particular because of the « end-point bias »: the last point of the series has an exaggerated impact on the trend at the end of the series. If one is only interested in the properties of the cycle, this is not that bad: one simply has to omit the trend values at the end of the series. But if the trend is used for economic policy, then the last point is likely to be the one which is particularly interesting.

The usual way[2] to solve this end-point bias problem is to extend the series (for example with ARIMA forecasts[3]). Thus the interesting point is no longer at the end of the series. The usefulness of this approach is limited, however, by the quality of the forecast[4]. We propose here an alternative which is robust in the sense that it does not use forecasts. It is a simple and natural modification of the HP filter.

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1 Hodrick, R.J. and E.C. Prescott (1997), « Postwar US Business Cycles: An Empirical Investigation », *Journal of Money, Credit and Banking*, 29, 1-16.

2 Kaiser R. & A. Maravall (2001), « Measuring Business Cycles in Economic Time Series », Lecture Notes in Statistics, Springer.

3 A sophisticated extrapolation process like the ARIMA forecast has the advantage of leaving no discretionary choice (or only at the beginning). It has however the disadvantage of not taking into account unexpected announcements (which have an impact on the future but are not taken into account in past data). This is why an expert forecast might be preferred to a pure ARIMA forecast.

4 If the series is extended by four years' forecasts (t+1 to t+4), and if there is an error in the level of these four forecasts, then (assuming that the error is the same for all four forecasts) it can be shown that more than 40% of this error will typically translate into error of the trend computed for time t (using HP with smoothing parameter 100). Some additional problems may appear if expert forecasts are used instead of ARIMA forecast. Expert forecasts of GDP growth rate for example might be biased toward the mean growth rate. This implies that forecast errors might systematically depend on business cycle conditions (being too pessimistic in boom times and too optimistic in recessions). It might not be possible to correct this systematic error in real time because it might be difficult to assess the business cycle conditions (or because one is computing the HP trend precisely in order to assess the business cycle conditions). Another problem with expert forecasts is that they leave a lot of freedom about the choice of a crucial value: how quickly the economy returns to normal after a shock.

Section 2 explains why the HP filter has an end-point bias, Section 3 proposes a modification of the HP filter, Section 4 describes some properties of the modified HP filter, and lastly Section 5 concludes.

2. Why does the HP filter have an end-point bias?

The HP filter defines the trend g_t such as to minimize the following penalty function:

$$\sum_{t=T-N+1}^{T} \frac{1}{\lambda}(y_t - g_t)^2 + \sum_{t=T-N+2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2 \tag{1}$$

where y_t is the value at time t of the variable for which we want to compute the trend g_t. N values of this variable are known up to point T in time. If there were only the first term, the solution would be $g_t = y_t$ at all time t and there would be no cyclical component. But the second term imposes a penalty on changes in the trend's slope. If λ were infinite, no change in the trend's slope would be allowed, and the trend would be a straight line (the same as the ordinary least squares regression line, where time would be the independent variable). The bigger λ is, the smoother the trend. This is why λ is called the « smoothing parameter ».

It can be seen at this point that the HP filter suffers from an end-point bias. The second term of the penalty function features a sum from t=T-N+2 to T-1 and not from t=T-N+1 to T as in the first term (if the sum in the second term were taken from t=T-N+1 to T there would be more unknown g_t than data y_t). The consequence is that g_{T-N+1}, g_{T-N+2}, g_{T-1} and g_T do not appear in the second term of (1) as often as the other g_t: g_{T-N+1} and g_T appear only one time, g_{T-N+2} and g_{T-1} appear two times, while all other g_t appear three times. Thus, for g_{T-N+1}, g_{T-N+2}, g_{T-1} and g_T the penalty for a trend kink is lower than it would be if all g were treated equally. When choosing g_{T-N+1}, g_{T-N+2}, g_{T-1} and g_T to minimize the penalty function, larger trend kinks will be allowed at the ends of the series than would be the case without that bias, resulting in data at the ends of the series having an exaggerated impact on the trend.

3. Proposition to modify the HP filter

The previous comments suggest an easy way to modify the HP filter in order to reduce the end-point bias: all g_t should appear in the same measure in the second term of the penalty function which penalizes changes in the trend's slope. This can be done by minimizing the following penalty function instead of (1):

$$\sum_{t=T-N+1}^{T} \frac{1}{\lambda_t}(y_t - \tilde{g}_t)^2 + \sum_{t=T-N+2}^{T-1}[(\tilde{g}_{t+1} - \tilde{g}_t) - (\tilde{g}_t - \tilde{g}_{t-1})]^2 \quad (2)$$

where

$\lambda_t = \lambda$ for t=T-N+3 to T-2

$\lambda_t = \lambda * 3/2$ for t=T-N+2 and t=T-1

$\lambda_t = \lambda * 3$ for t=T-N+1 and t=T

The idea is to compensate for the fact that for certain values of t, $\tilde{g}_t$ appears less often in the second term of the penalty function, by increasing the corresponding value of λ (for example $\tilde{g}_T$ appears only once instead of three times, thus its λ_T is three times the usual λ). This modification is not enough to make all $\tilde{g}_t$ enter symmetrically into the penalty function, but it makes it more symmetrical in the sense that a change in the trend's slope always costs three penalty terms. Thus it does not completely solve the end-point bias, but it reduces it. This modification is quite natural and does not lead to a situation in which the λs near the end of the series could be set at discretion. But the proof of the pudding is in the eating, and the main argument in favor of the modified HP filter is its properties.

4. Some properties of the modified HP filter

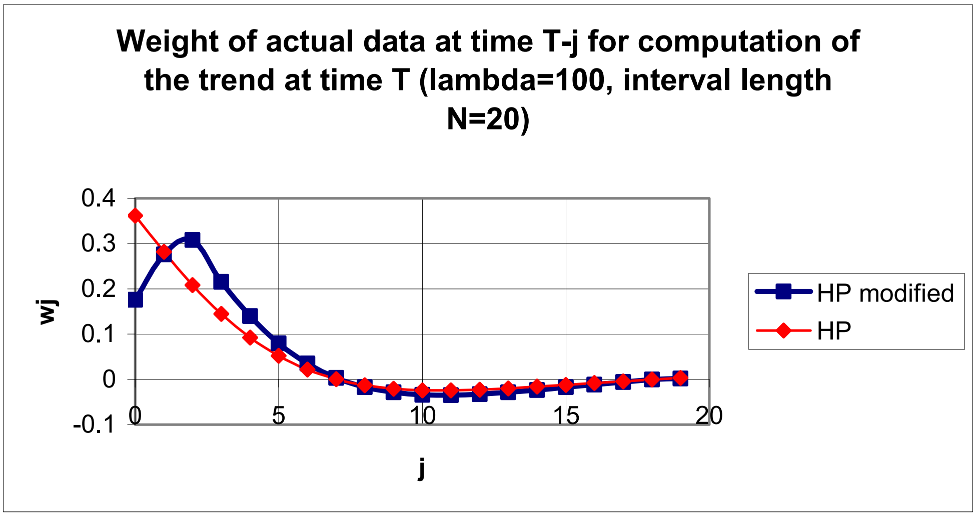
Applied recursively, the modified HP filter has the three following advantages compared to the usual HP filter: the end-point bias is reduced (§4.1), the business cycle component (difference between actual data and trend) is bigger (§4.2), cycles at business cycles frequencies are eliminated from the trend to a greater extent (§4.3).

4.1. The end-point bias is reduced

The trend can be expressed as a linear combination of the data of the initial series:

$$g_t = \sum_{j=0}^{N-1} w_j y_{T-j} \tag{3}$$

The following graph shows the weights w_j (vertical axis) for all j (horizontal axis) from 0 to N-1 (for λ=100 and N=20).



We can see that a revision of the data at time T will have less effect on the trend at time T with the modified HP filter than with the usual HP filter. This reduces the end-point bias[5]. The impact is the same for the data at T-1, and then greater for several years[6].

The fact that the modified HP filter gives more weight than the HP filter to the data of the past is the counterpart to the fact that the last point has less impact on the trend. A monotone curve for the weights of the modified HP filter might have been more satisfying than this shape[7] with its maximum for data at t-2. However, this maximum has the desirable property of being below the maximum of the HP weights.

The previous graph is made for recursive application of the filters on an interval of length 20, but when λ=100 the length of this interval is not important so long as it is greater than 20. For the modified HP filter this can be seen on the following graph (the usual HP filter has the same property), which shows the weights for the modified HP filter applied to intervals of various lengths.

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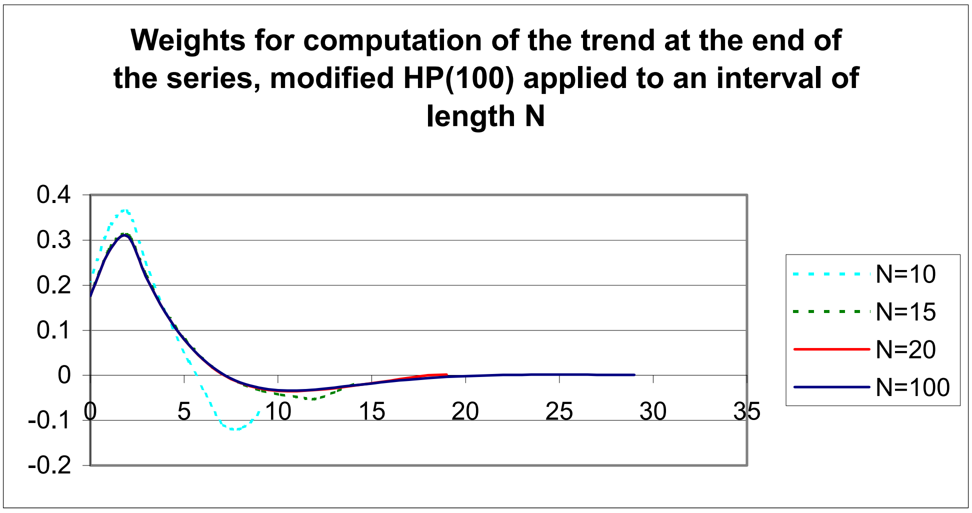
[5] The end-point problem is reduced in the sense that the excess weight of the last data is reduced. However, it does not follow from this that the difference between the ex-post and recursive trend is necessarily smaller with the modified HP filter than with the usual HP. I am indebted to Yvan Lengwiler who made me aware of this point (he showed an example with US data in which the difference between recursive and ex-post trend is larger with the modified HP filter).

[6] The sum of the weights is equal to 1, but some weights are negative. This is true for the HP filter as well as for the modified HP filter. This would also be true for an ordinary least square regression of the data on time. The simplest way to understand why some weights are negative is to consider an ordinary least square regression (with time as independent variable) in a case in which the slope of the regression line is positive and the scatter contains many points. An increase (small enough such that the center of gravity of the scatter, through which the regression line must pass, be approximately constant) of the data near the end of the time regression interval will increase the trend at the end of the interval, while an increase of the data near the beginning of the interval will decrease the trend at the end of the interval (because it will reduce the slope of the regression line). Thus, it is justified that some weights are negative. The fact that the negative weights are more negative with the modified HP should not a priori be seen as an undesirable feature of this filter.

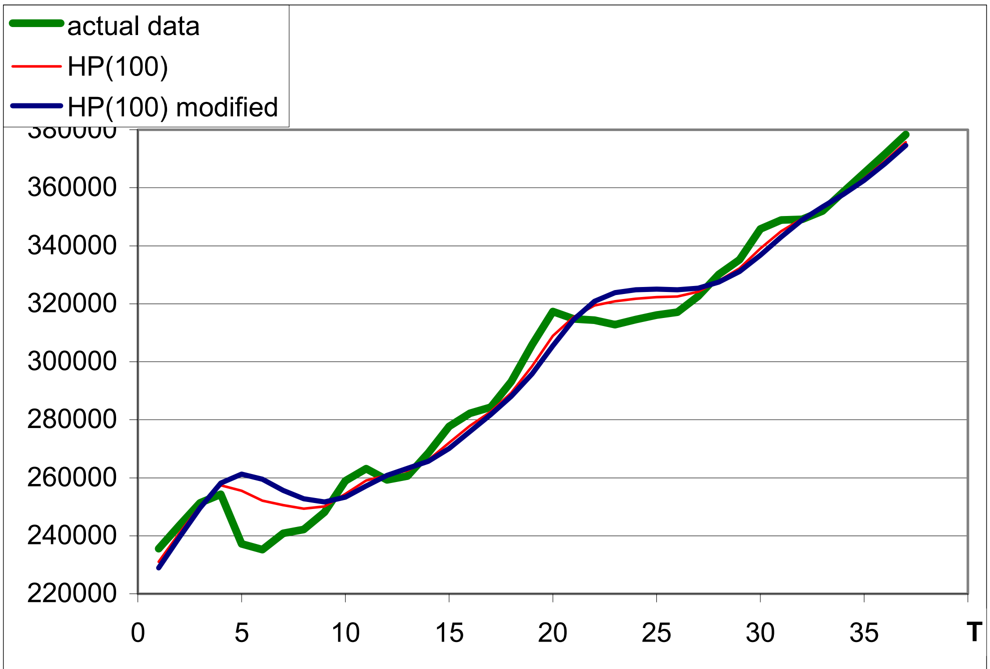
[7] However, this shape might be appreciated for practical reasons not discussed here: old data should have little weight because they have little impact on current data, and very recent data should have less weight because they might be revised; this leads to a shape featuring a maximum.



We can see that these weights are almost identical for N=20 and N=100 (in fact the two curves are so similar that they cannot be distinguished over the area in which they are both defined).

4.2. The business cycle component is bigger

The following graph shows an example of data (actually this data is a real GDP series) and their trend computed recursively with the HP filter and the modified HP filter (the trend at time T is computed by applying the filter to an interval of length N=20 finishing at T).



Filters recursively applied on an interval of length 20

We can see that according to the HP filter the trend is (almost) always situated between the actual data curve and the trend according to the modified HP filter. This means that compared to the HP filter, the modified HP filter attributes a greater portion of the fluctuations to the business cycle.

The fact that the data, the trend according to the HP filter, and the trend according to the modified HP filter intersect at the same point may appear surprising. In fact, they do not cross precisely at the same point, but nearly. The intuitive explanation of this phenomenon is simple. Suppose that at a certain date the trend according to the modified filter is equal to the actual data. This implies that in the penalty function the value of $1/\lambda_T$ is not relevant because it is multiplied by $(y_T - g_T)^2$ which is null. The fact that the value of λ at time T has been tripled (in comparison to the usual HP filter) has no effect. Furthermore, $g_T=y_T$ implies that $[(g_T - g_{T-1}) - (g_{T-1} - g_{T-2})]^2=0$. Indeed, if $[(g_T - g_{T-1}) - (g_{T-1} - g_{T-2})]^2$ were different from 0, it would be preferable to modify g_T in order to reduce the value of this expression, even if that means increasing $(y_T - g_T)^2$: the derivative of $(y_T - g_T)^2$ with respect to g_T being null when $(y_T - g_T)$=0, it would not cost much to increase this value slightly. But if $[(g_T - g_{T-1}) - (g_{T-1} - g_{T-2})]^2$=0, that means that having multiplied λ by 3/2 at time T-1 had relatively little impact. Indeed, if one extracts the terms of the penalty function containing g_{T-1}, one obtains (after having multiplied them by λ_{T-1}):

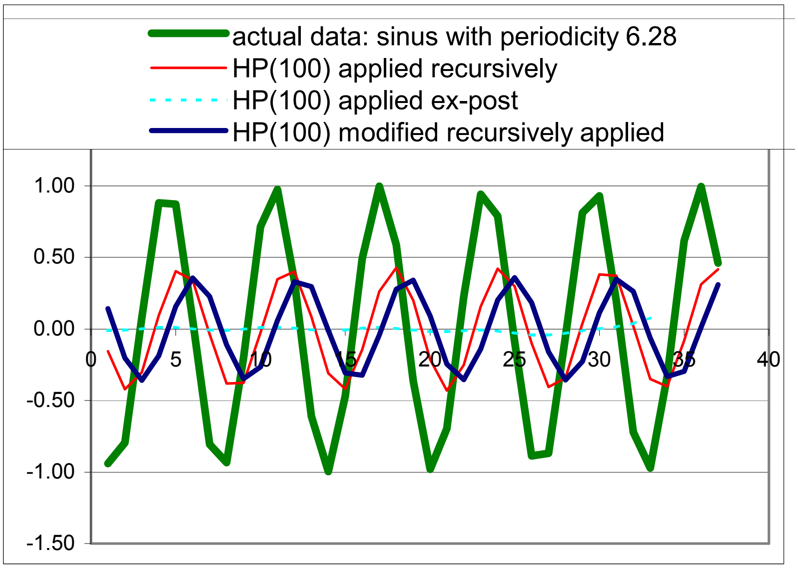
$$(y_{T-1} - g_{T-1})^2 + \lambda_{T-1}\left\{[(g_T - g_{T-1}) - (g_{T-1} - g_{T-2})]^2 + [(g_{T-1} - g_{T-2}) - (g_{T-2} - g_{t-3})]^2\right\}$$

The fact that $[(g_T - g_{T-1}) - (g_{T-1} - g_{T-2})]^2$=0 makes the term which is multiplied by λ_{T-1} diminish toward zero, and thus reduces the impact of λ_{T-1}, and thereby the effect of having multiplied this coefficient by 3/2. Conclusion: the modified HP filter differs from the usual HP filter by the values of λ at T and at T-1, but if g_T=y_T then the value of λ at point T has no effect and the value of λ at point T-1 has less effect than usual. So, if at a given date the trend according to the modified HP filter is equal to the actual data, it is normal that the trend according to the usual HP filter also be approximately (but not exactly) equal to the actual data.

It is shown in the appendix that it is possible to be more precise than just saying that the business cycle component is larger with the modified HP filter: it is 43% larger (for λ=100).

4.3. Business cycle fluctuations are attenuated to a greater extent (but there is a phase shift)

The following graph shows the trend of an artificial series, sin(t), computed with the HP filter as well as the trend obtained with the modified HP filter. The trend computed with the HP filter applied ex-post to the entire series is also shown (this trend is not drawn for the last four years of the series).



We can see that the amplitude and the periodicity of the oscillations are approximately the same for the HP filter and for the modified HP filter. They both attenuate these oscillations, as desired since a cycle over a six-year period can be considered to be a business cycle (remember that these filters compute a trend which should not contain any business cycle fluctuations). The trend according to the modified HP filter is slightly out of phase with respect to the HP filter (the difference is small compared to the periodicity). Intuitively, this reflects the fact that the modified HP filter gives less weight to the end-point of the series (that was the goal because this weight is exaggerated in the HP filter) and therefore more weight to the past data. Some inertia results from this[8].

What happens if we take a sinus for another period? The following graph provides an example computed with a sinus with a longer periodicity.

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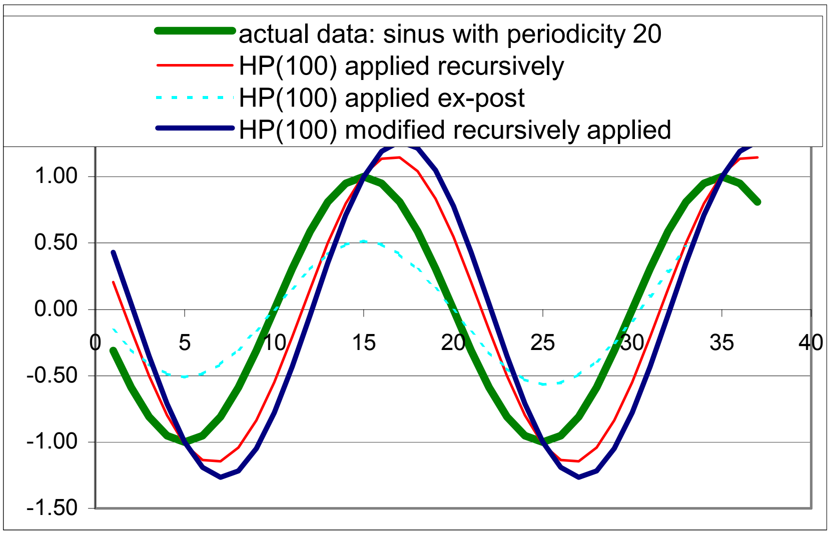
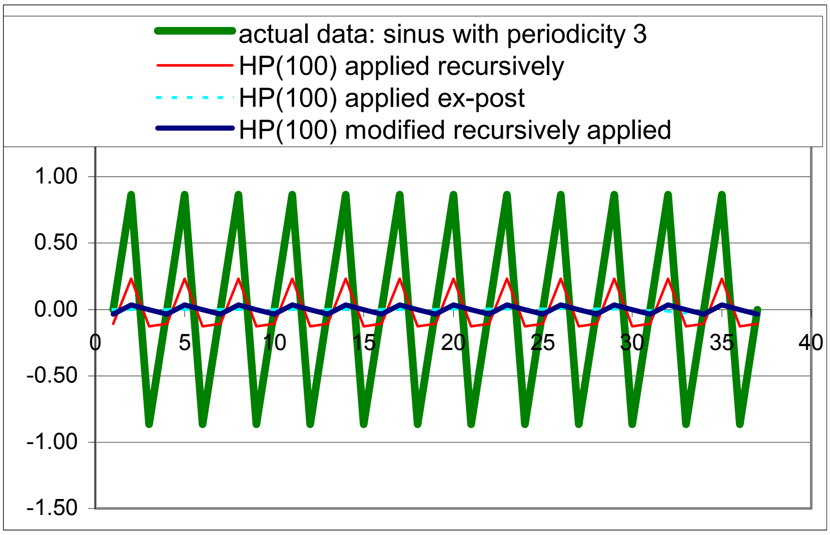
[8] There is a trade-off between continuing to reduce a sinus of a given period by the same amount, avoiding phase shift and eliminating the end-point bias. See Schips Bernard, « Einige Anmerkungen zur „Saisonbereinigung“ von Zeitreihen », KOF Konjunktur Bericht 2/2003.



Here the trend according to the modified HP filter is still out of phase relative to the HP filter, and the amplitudes are similar. The fluctuations are not attenuated. This is correct since a cycle over a twenty-year period is not a business cycle.

The amplitudes can differ to a greater extent if the periodicity is weaker, as the following graph shows:



Therefore, it becomes clear that the modified HP filter has the advantage of attenuating high frequency fluctuations to a greater extent than the usual HP filter. The amplitude according to the modified HP filter becomes closer to the one that would have been obtained by applying the usual HP filter ex-post. This however was not the case for oscillations of large periodicity. Thus the usual HP filter applied recursively can be closer to the ex-post HP filter than the modified HP filter (also applied recursively) if the data contain enough oscillations of large periodicity, which can be the case for non-stationary data.

If it is possible to make accurate forecasts, it is preferable to extend the series with these forecasts rather than using the modified HP filter: in this way we get closer to trend according to the usual HP filter ex-post. However, the modified HP filter has the advantage of being robust in the sense that it does not depend upon forecasts and therefore does not depend upon their quality.

Following the spectral analysis approach we can compute analytically the impact of the two filters on $y_t=\sin(t*2\pi/\tau)$, a sinus of period τ and frequency $2\pi/\tau$. Let $g_t = \sum_{j=0}^{N-1} w_j y_{t-j}$ be a linear filter where w_j are the weights (remember that both the usual HP and the modified HP are filters of this type). Applying this filter to the sinus will give the following trend: $g_t = \sum_{j=0}^{N-1} w_j \sin[(t - j) * 2\pi/\tau]$.

Using the identity $\sin(\theta+\varphi)=\sin(\theta)\cos(\varphi)+\cos(\theta)\sin(\varphi)$, this can be written:

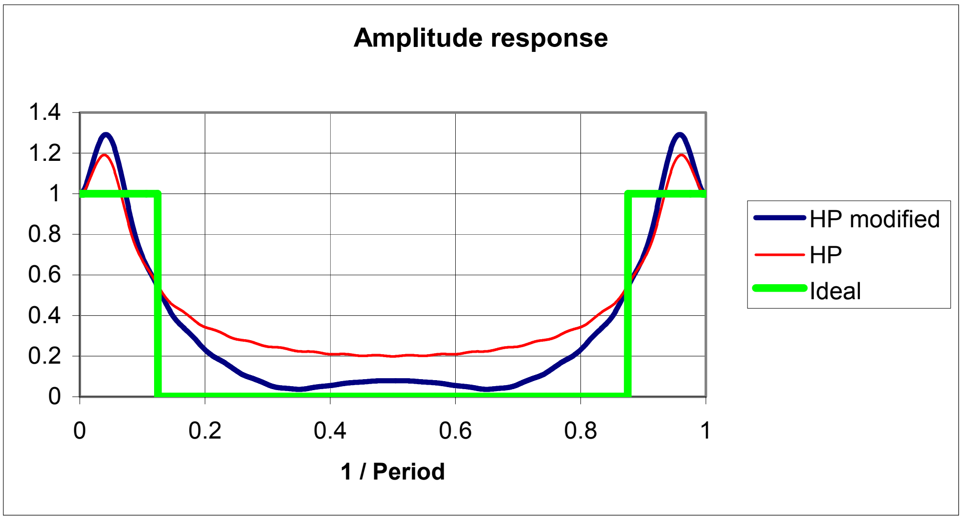
$$g_t = \sin(t * 2\pi/\tau)[\sum_{j=0}^{N-1} w_j \cos(-j * 2\pi/\tau)] + \cos(t * 2\pi/\tau)[\sum_{j=0}^{N-1} w_j \sin(-j * 2\pi/\tau)] \qquad (4)$$

$$= \sin(t * 2\pi/\tau + \Omega)\sqrt{[\sum_{j=0}^{N-1} w_j \cos(-j * 2\pi/\tau)]^2 + [\sum_{j=0}^{N-1} w_j \sin(-j * 2\pi/\tau)]^2}$$

where $\Omega = arctg(\frac{\sum_{j=0}^{N-1} w_j \sin(-j * 2\pi/\tau)}{\sum_{j=0}^{N-1} w_j \cos(-j * 2\pi/\tau)]})$

This is a sinusoid of amplitude $\sqrt{[\sum_{j=0}^{N-1} w_j \cos(-j * 2\pi/\tau)]^2 + [\sum_{j=0}^{N-1} w_j \sin(-j * 2\pi/\tau)]^2}$ and phase shift Ω.

The original sinus had amplitude of 1, thus this formula gives the amplitude response of the filter. Since we know the values of the weights w_j for the usual HP and the modified HP filters, we can compute the amplitude response for various frequencies. For the HP, the modified HP and an ideal filter (defined as one which eliminates all cycles of period equal or smaller than 8 years, and leaves the other cycles unchanged), the following graph shows the amplitude response on the vertical axis as a function of the frequency (expressed as a fraction of 2π, thus equal to $1/\tau$) indicated on the horizontal axis (we do not consider periods smaller than the time unit).



The ideal filter keeps only low frequency oscillations (as well as very high frequency oscillations which are close enough to the time unit that they are observationally equivalent to low frequency oscillations). The amplitude response of the modified HP filter is often closer to the ideal than the one of the HP filter (the fact that the three curves cross at the same point is a desirable, but unexpected, feature).

The following graph shows the phase shift[9] (the optimal phase shift is null):

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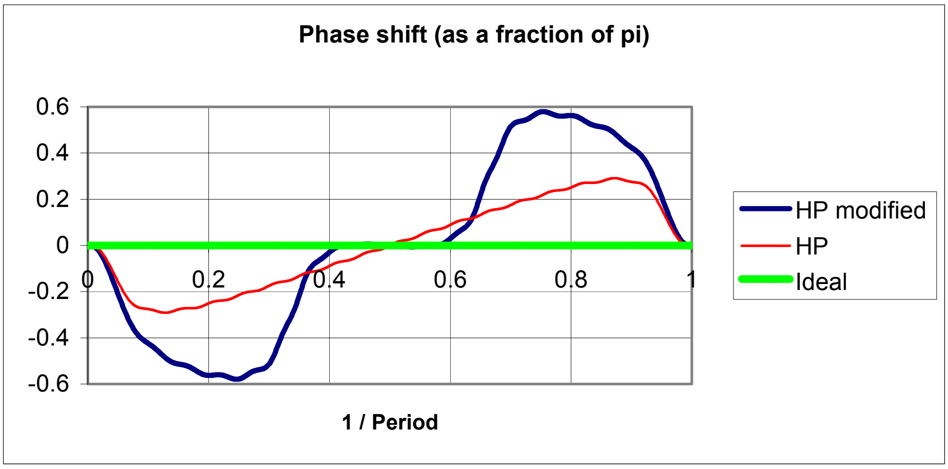
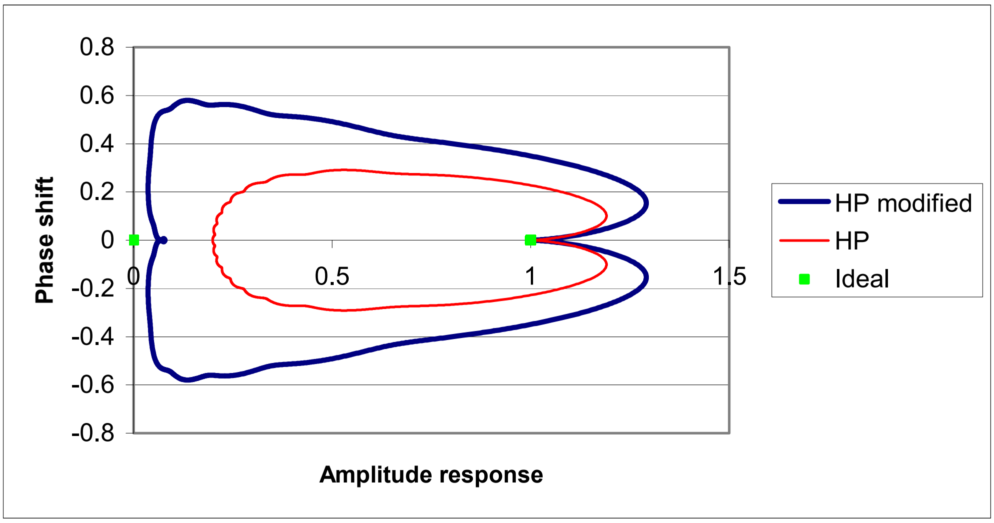
[9] It is well known that the graphs for the amplitude response and the phase shift feature a symmetry: the amplitude response for $1-1/\tau$ is the same as for $1/\tau$, and the phase shift is the



The phase shift is usually worse for the modified HP filter. Luckily, the increase of phase shift is the largest for frequencies which are strongly attenuated:



5. Conclusion

A modified version of the HP filter is proposed which makes it possible to reduce the end-point bias at the cost of a phase shift. The usual approach consisting in extending the series with forecast values is preferable when these forecasts are

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opposite. To see this, plug $1\text{-}1/\tau$ in place of $1/\tau$ into equation (4) and notice that $\Sigma w_j\cos(-j2\pi(1-1/\tau))= \Sigma w_j\cos(-j2\pi 1/\tau)$ and $\Sigma w_j\sin(-j2\pi(1-1/\tau))= -\Sigma w_j\sin(-j2\pi 1/\tau)$.

accurate. However, the approach presented here has the advantage of being more robust in the sense that it is independent of the quality of the forecasts. It might be particularly useful for recursive application on yearly GDP data since the GDP forecast for several years ahead is unlikely to be accurate.

Appendix

Equation (3) can be rewritten as:

$$
\begin{aligned}
g_t &= \sum_{j=0}^{N} w_j y_{t-j} \\
&= w_0 y_t + \left[\sum_{j=1}^{N-1} w_j y_{t-j}\right] + w_N y_{t-N} \\
&= \left(\sum_{i=0}^{N} w_i - \sum_{i=1}^{N} w_i\right) y_t + \sum_{j=1}^{N-1}\left[\left(\sum_{i=j}^{N} w_i - \sum_{i=j+1}^{N} w_i\right) y_{t-j}\right] - \left[\left(-\sum_{i=N}^{N} w_i\right) y_{t-N}\right] \\
&= \left(\sum_{i=0}^{N} w_i - \sum_{i=1}^{N} w_i\right) y_t + \sum_{j=1}^{N-1}\left[\left(-\sum_{i=j+1}^{N} w_i\right) y_{t-j}\right] - \sum_{j=1}^{N-1}\left[\left(-\sum_{i=j}^{N} w_i\right) y_{t-j}\right] - \left[\left(-\sum_{i=N}^{N} w_i\right) y_{t-N}\right] \\
&= \left(\sum_{i=0}^{N} w_i\right) y_t + \sum_{j=0}^{N-1}\left[\left(-\sum_{i=j+1}^{N} w_i\right) y_{t-j}\right] - \sum_{j=1}^{N}\left[\left(-\sum_{i=j}^{N} w_i\right) y_{t-j}\right] \\
&= y_t + \sum_{j=0}^{N-1}\left[\left(-\sum_{i=j+1}^{N} w_i\right) y_{t-j}\right] - \sum_{j=0}^{N-1}\left[\left(-\sum_{i=j+1}^{N} w_i\right) y_{t-j-1}\right] \\
&= y_t + \sum_{j=0}^{N-1}\left[\left(-\sum_{i=j+1}^{N} w_i\right)\left(y_{t-j} - y_{t-j-1}\right)\right]
\end{aligned}
$$

Thus $y_t - g_t = \sum_{j=0}^{N-1}\left[\tilde{w}_j\left(y_{t-j} - y_{t-j-1}\right)\right]$ where $\tilde{w}_j = \sum_{i=j+1}^{N} w_i$ (4)

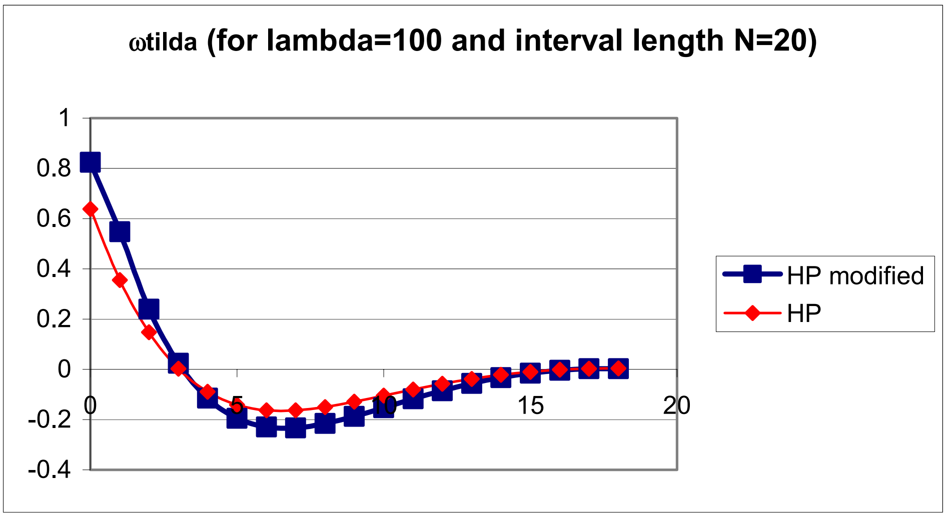
Notice that $\sum_{j=0}^{N-1} \tilde{w}_j = \sum_{j=0}^{N-1}\left[\sum_{i=j+1}^{N} w_i\right] = \sum_{j=0}^{N} j w_j = 0$

The last equality follows from the assumption that the filter leaves a straight line y_t=a+bt unchanged.

$$
a + bt = y_t = g_t = \sum_{j=0}^{N} w_j y_{t-j} = \sum_{j=0}^{N} w_j \left(a + b(t-j)\right) = (a+bt)\left(\sum_{j=0}^{N} w_j\right) - b\sum_{j=0}^{N} w_j j = (a+bt) - b\sum_{j=0}^{N} w_j j
$$

Thus $\sum_{j=0}^{N} j w_j = 0$

This means that the difference between the actual data and the trend can be written as a linear combination of the increases in y from one period to the next. The following graph shows the weights $\tilde{w}_j$



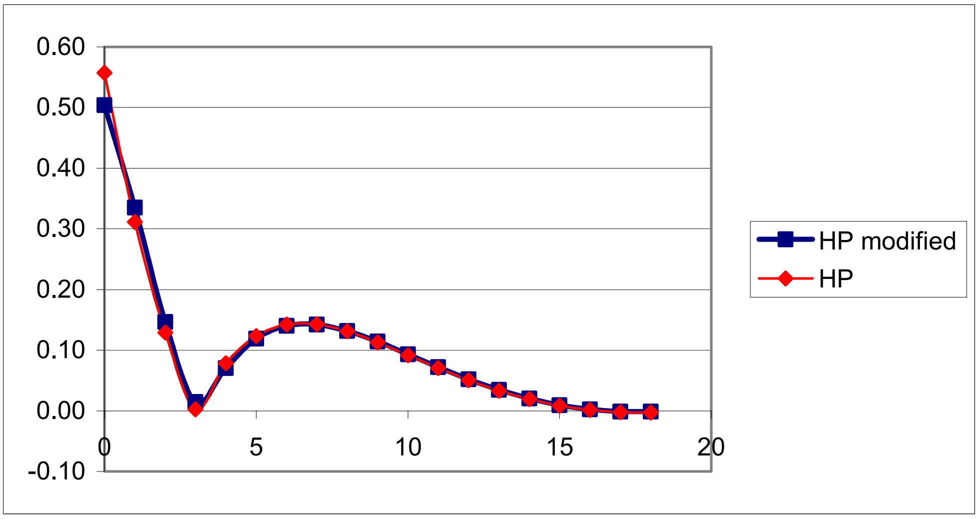
Since $\tilde{w}_j$ is positive for j<4 we can write (for HP and modified HP):

$$y_t - g_t = \sum_{j=0}^{3}\left[\tilde{w}_j\left(y_{t-j} - y_{t-j-1}\right)\right] + \sum_{j=4}^{N-1}\left[\tilde{w}_j\left(y_{t-j} - y_{t-j-1}\right)\right]$$

$$= \left(\sum_{j=0}^{3}\tilde{w}_j\right)\left\{\sum_{j=0}^{3}\left[\frac{\tilde{w}_j}{\sum_{j=0}^{3}\tilde{w}_j}\left(y_{t-j} - y_{t-j-1}\right)\right] - \sum_{j=4}^{N-1}\left[\frac{\tilde{w}_j}{\sum_{j=4}^{N-1}\tilde{w}_j}\left(y_{t-j} - y_{t-j-1}\right)\right]\right\}$$

This means that the difference between the actual data and the trend is proportional to the difference between a weighted average (positive weights, sum equal to 1) of the yearly (assuming that the time unit is the year) changes of the four most recent dates, and a weighted average (positive or very close to 0, sum=1) of the previous yearly changes. Thus, the trend will be below the actual data if the recent yearly changes of the data are on (weighted) average larger than changes in previous years.

These weights $\frac{\tilde{w}_j}{\sum_{j=0}^{3}\tilde{w}_j}$ for j=0 to 3 and $\frac{\tilde{w}_j}{\sum_{j=4}^{N-1}\tilde{w}_j}$ for j=4 to N-1 are fairly similar (although not exactly equal) for the HP filter and the modified HP filter:



In first approximation, the major difference between the HP filter and the modified HP filter is the proportionality coefficient $\sum_{j=0}^{3}\tilde{w}_j$ which is 1.64 for the modified HP and 1.14 for HP. This means that in first approximation the main difference between HP(100) and modified HP(100) is that the gap y_t-g_t is 43% larger for the modified HP. This confirms our result of section 4.2, with the additional information that the business cycle component is in first approximation always 43% larger for the modified HP.

The approximation $y_{tW} - g_t^{MHP} \approx \frac{\sum_{j=0}^{3}\tilde{w}_j^{MHP}}{\sum_{j=0}^{3}\tilde{w}_j^{HP}}\left(y_t - g_t^{HP}\right) = \alpha * \left(y_t - g_t^{HP}\right)$ with α=1.43 can be used to derive some properties of the modified HP filter. For example it implies that

$$g_t^{HP} \approx \frac{1}{\alpha} g_t^{MHP} + \left(1 - \frac{1}{\alpha}\right) y_t$$

Thus the trend according to HP is a weighted average of the actual data y_t and the trend according to the modified HP. Further properties can be computed. For example, the spectral properties of the modified HP filter can be computed as a function of the spectral properties of the HP filter:

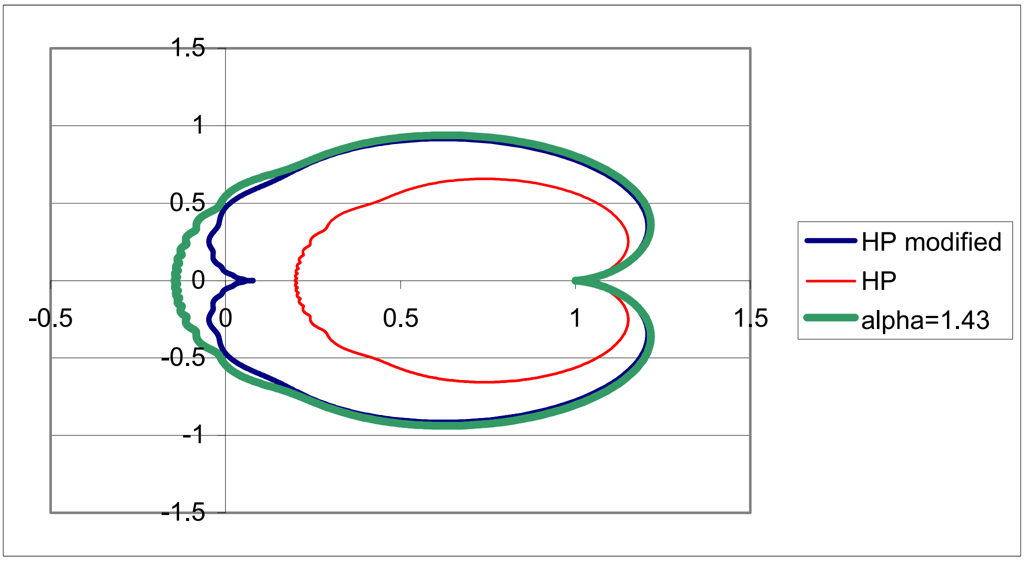
If $y_t=e^{it2\pi/\tau}$ then $g_t^{HP} = A_{HP} e^{i\left(t\frac{2\pi}{\tau}+\varphi_{HP}\right)}$

$$g_t^{MHP} \approx \alpha g_t^{HP} + (1-\alpha) y_t = \alpha A_{HP} e^{i\left(t\frac{2\pi}{\tau}+\varphi_{HP}\right)} + (1-\alpha) e^{it\frac{2\pi}{\tau}} = \left[\alpha A_{HP} e^{i\varphi_{HP}} + (1-\alpha)\right] e^{it\frac{2\pi}{\tau}}$$
$$= \left|\alpha A_{HP} e^{i\varphi_{HP}} + (1-\alpha)\right| e^{i\left(t\frac{2\pi}{\tau}+\arg(\alpha A_{HP} e^{i\varphi HP} + (1-\alpha))\right)}$$

Thus,

$$A_{MHP} = \sqrt{[\alpha A_{HP}\cos(\varphi_{HP}) + (1-\alpha)]^2 + [\alpha A_{HP}\sin(\varphi_{HP})]^2} = \sqrt{(1-\alpha)^2 + 2(1-\alpha)\alpha A_{HP}\cos(\varphi_{HP}) + (\alpha A_{HP})^2}$$
$$tg(\varphi_{MHP}) = \frac{\alpha A_{HP}\sin(\varphi_{HP})}{\alpha A_{HP}\cos(\varphi_{HP}) + (1-\alpha)}$$

The following graph presents g_t^{HP}, g_t^{MHP} and $g_t^{computed\ from\ HP\ with\ \alpha=1.43}$ in the complex space:



We see that the spectral properties of the filter computed from HP corrected with α=1.43 are usually very close to the one of the HP modified.